3 Singularities

3.1 Zeros

Definition 3.1. Let $D \subseteq \mathbb{C}$ be a region, $z_0 \in D$ and $f \in \mathcal{O}(D)$ such that $f(z_0) = 0$. We say that f has a *zero of order* n at z_0 iff there exists $g \in \mathcal{O}(D)$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^n g(z)$ for all $z \in D$.

Proposition 3.2. *Let $D \subseteq \mathbb{C}$ be a region, $z_0 \in D$ and $f \in \mathcal{O}(D)$ such that $f(z_0) = 0$. If f is not constant, then there exists a unique $n \in \mathbb{N}$ such that f has a zero of order n at z_0. Moreover, $n = \inf\{k \in \mathbb{N} : f^{(k)}(z_0) \neq 0\}$.*

Proof. **Exercise**. □

Proposition 3.3 (Fundamental Theorem of Algebra)**.** *Let $n \in \mathbb{N}$ and $p(z) = \sum_{k=0}^{n} c_k z^k$ be a polynomial of degree n (i.e., $c_n \neq 0$). Then, there are constants $a_1, \ldots, a_n \in \mathbb{C}$ such that p factorizes as*

$$p(z) = c_n(z - a_1) \cdots (z - a_n).$$

Proof. **Exercise**.[Hint: First show the existence of one zero and factorize it, then proceed recursively.] □

Theorem 3.4. *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{O}(D)$ such that it has distinct zeros $a_1, \ldots, a_m \in D$ with orders $n_1, \ldots, n_m$. Suppose γ is a closed path in $D \setminus \{a_1, \ldots, a_n\}$ such that $\mathrm{Int}_\gamma \subset D$. Then,*

$$\sum_{k=1}^{m} n_k \mathrm{Ind}_\gamma(a_k) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z.$$

Proof. Knowing the zeros, we can factorize f as

$$f(z) = (z - a_1)^{n_1} \cdots (z - a_m)^{n_m} g(z),$$

where $g \in \mathcal{O}(D)$ has no zeros in D. Using the product rule for the derivative we find,

$$\frac{f'(z)}{f(z)} = \frac{g'(z)}{g(z)} + \sum_{k=1}^{m} \frac{n_k}{z - a_k}.$$

The term g'/g on the right hand side is a holomorphic function in D. So, by Theorem 2.43 its integral along γ vanishes. The second term yields the desired sum over the indices of the a_k. □

Exercise 27. Let $D \subseteq \mathbb{C}$ be a region and $a \in D$. For a function $f \in \mathcal{O}(D)$ we denote by $n_a(f)$ the order of its zero at a. (If $f(a) \neq 0$ then $n_a(f) = 0$.) For all $f, g \in \mathcal{O}(D)$ show the following:

1. $n_a(fg) = n_a(f) + n_a(g)$.
2. $n_a(f+g) \geq \min\{n_a(f), n_a(g)\}$ and equality if $n_a(f) \neq n_a(g)$.

3.2 Singularities

Definition 3.5. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Then, we say that f has an *isolated singularity* at a. Moreover, a is called a *removable singularity* iff f can be extended to a holomorphic function on all of D.

We have already seen criteria for identifying removable singularities in the Riemann Continuation Theorem (Theorem 2.28).

Definition 3.6. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. We say that a is a *pole* of f iff f diverges at a, i.e. if for any $M > 0$ there exists $r > 0$ such that $|f(z)| > M$ for all $z \in B_r(a) \setminus \{a\}$. We say that a is an *essential singularity* of f iff a is not removable and is not a pole.

We now consider poles.

Proposition 3.7. *Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Suppose that a is a pole of f. Then, there exists a unique $m \in \mathbb{N}$ such that there is a $g \in \mathcal{O}(D)$ with $g(a) \neq 0$ and*

$$f(z) = \frac{g(z)}{(z-a)^m} \quad \forall z \in D \setminus \{a\}.$$

Proof. Since f has a pole at a there exist $r > 0$ such that $B_r(a) \subseteq D$ and $f(z) \neq 0$ for all $z \in B_r(a) \setminus \{a\}$. Thus we can define $h \in \mathcal{O}(B_r(a) \setminus \{a\})$ by $h(z) := 1/f(z)$. But $\lim_{z \to a} h(z) = 0$, so by Theorem 2.28, a is a removable singularity of h and h can be extended to a holomorphic function on all of $B_r(a)$. By Proposition 3.2 there exists a unique $m \in \mathbb{N}$ such that $h(z) = (z-a)^m k(z)$, where $k \in \mathcal{O}(B_r(a))$ and $k(a) \neq 0$. Moreover, $k(z) \neq 0$ for all $z \in B_r(a)$ so we can invert it, defining $g \in \mathcal{O}(B_r(a))$ by $g(z) = 1/k(z)$. But notice that $g(z) = (z-a)^m f(z)$ for all $z \in B_r(a) \setminus \{a\}$, which obviously extends to a holomorphic function on $D \setminus \{a\}$. So g really extends to a holomorphic function on all of D. Observe also that $g(a) \neq 0$. This completes the proof. $\square$

Definition 3.8. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$, $f \in \mathcal{O}(D \setminus \{a\})$ such that a is a pole of f. Then, the integer $m \in \mathbb{N}$ such that $g(z) := (z-a)^m f(z)$ extends to a holomorphic function in D with $g(a) \neq 0$ is called the *order* of the pole. If $m = 1$ we also say that the pole is *simple*.

Proposition 3.9. *Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$ with a pole at a of order m. Then, is a function $g \in \mathcal{O}(D)$ and there are constants $b_1, \ldots, b_m \in \mathbb{C}$ with $b_m \neq 0$ such that*

$$f(z) = g(z) + \sum_{n=1}^{m} \frac{b_n}{(z-a)^n} \quad \forall z \in D \setminus \{a\}.$$

Proof. **Exercise.** □

The second term on the right hand side of the equation above is also called the *singular part* of f at a.

We now turn to essential singularities. In some sense they are more "wild" than poles, as shows the following Theorem.

Theorem 3.10 (Casorati, Weiserstrass)**.** *Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. The following statements are equivalent:*

1. *The point a is an essential singularity of f.*
2. *For every neighborhood $U \subseteq D$ of a the set $f(U \setminus \{a\})$ is dense in $\mathbb{C}$.*
3. *There exists a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $D \setminus \{a\}$ such that $\lim_{n \to \infty} z_n = a$, but $\{f(z_n)\}_{n \in \mathbb{N}}$ has no limit in $\mathbb{C} \cup \{\infty\}$.*

Proof. We start with the implication 1.⇒2. Assume the contrary of 2. Let $U \subseteq D$ be a neighborhood of a such that $f(U \setminus \{a\})$ is not dense in $\mathbb{C}$. Thus, there exists $p \in \mathbb{C}$ and $r > 0$ such that $f(U \setminus \{a\}) \cap B_r(p) = \emptyset$. This implies $|f(z) - p| \geq r$ for all $z \in U \setminus \{a\}$. Define $g \in \mathcal{O}(U \setminus \{a\})$ by $g(z) := 1/(f(z) - p)$. Then, $|g(z)| \leq 1/r$ for all $z \in U \setminus \{a\}$ so by Theorem 2.28, g has a removable singularity at a. Thus, $c := \lim_{z \to a} g(z)$ exists. If $c \neq 0$, $f(z) = q + 1/g(z)$ is bounded near a and thus has a removable singularity at a. If $c = 0$, then $\lim_{z \to a} |f(z)| = \infty$ and f has a pole at a. In both cases, a is not an essential singularity, contradicting 1. **Exercise.**Complete the proof. □

Exercise 28. Find and classify the isolated singularities of the following functions and specify the order in case of a pole:

1. $\dfrac{z^4}{(z^4+16)^2}$
2. $\dfrac{1-\cos(z)}{\sin z}$
3. $\exp(1/z)$
4. $\dfrac{1}{\cos(1/z)}$

Exercise 29. Let f be a function that is holomorphic on $\mathbb{C}$ except for poles. Show that the set of poles cannot have an accumulation point.

Exercise 30. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Show that if a is a non-removable singularity of f, then $\exp \circ f \in \mathcal{O}(D \setminus \{a\})$ has an essential singularity at a.

Exercise 31. Let $D \subseteq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Let $P \in \mathcal{O}(\mathbb{C})$ be a non-constant polynomial. Show that f and $P \circ f$ have the same type of singularity at a.

3.3 Laurent Series

The representation of a holomorphic function with a pole as in Proposition 3.9 can be written as an "extended" power series that starts not with the power 0, but with the power $-n$. Indeed, we will see that even essential singularities can be captured by such an "extended" power series, if we start at $-\infty$. Such series are called *Laurent series*.

Let $z \in \mathbb{C}$ and $0 < r_1 < r_2$. In the following we use the notation

$$A_{r_1,r_2}(z) := B_{r_2}(z) \setminus \overline{B_{r_1}(z)}.$$

This type of region is called an (open) annulus. Note the special case of the punctured disk $A_{0,r}(z) = B_r(z) \setminus \{z\}$.

Definition 3.11. Let $\{a_n\}_{n\in\mathbb{Z}}$ be an indexed set of complex numbers. We say that $\sum_{n\in\mathbb{Z}} a_n$ *converges (absolutely)* iff $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=1}^{\infty} a_{-n}$ both converge (absolutely). Let S be a set and $\{f_n\}_{n\in\mathbb{Z}}$ be an indexed set of functions $f_n : S \to \mathbb{C}$. We say that $\sum_{n\in\mathbb{Z}} f_n$ converges uniformly iff $\sum_{n=0}^{\infty} f_n$ and $\sum_{n=1}^{\infty} f_{-n}$ both converge uniformly.

Proposition 3.12. *Let $\{c_n\}_{n\in\mathbb{Z}}$ be an indexed set of complex numbers. Define $r_1, r_2 \in [0,\infty]$ via*

$$r_1 := \limsup_{n\to\infty} |c_{-n}|^{1/n} \quad \textit{and} \quad 1/r_2 := \limsup_{n\to\infty} |c_n|^{1/n}.$$

Iff $r_1 < r_2$ then the Laurent series

$$f(z) = \sum_{n\in\mathbb{Z}} c_n z^n$$

converges absolutely for all $z \in A_{r_1,r_2}(0)$ and uniformly on $A_{\rho_1,\rho_2}(0)$ where $r_1 < \rho_1 < \rho_2 < r_2$. Moreover, it diverges for $z \in \mathbb{C} \setminus \overline{A_{r_1,r_2}(0)}$.

Proof. **Exercise.**[Hint: Split the series into the parts with positive and negative indices and apply Lemma 1.14.] □

Proposition 3.13. *Let $D \subseteq \mathbb{C}$ be a region, $z_0 \in \mathbb{C}$ and $0 \le r_1 < r_2$ such that $\overline{A_{r_1,r_2}(z_0)} \subset D$. Then, for all $f \in \mathcal{O}(D)$ we have,*

$$\int_{\partial B_{r_1}(z_0)} f = \int_{\partial B_{r_2}(z_0)} f.$$

Moreover, for all $z \in A_{r_1,r_2}(z_0)$ we have,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{r_2}(z_0)} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{r_1}(z_0)} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta$$

Proof. **Exercise.** □

Theorem 3.14 (Laurent Decomposition)**.** *Let $z_0 \in \mathbb{C}$ and $0 \le r_1 < r_2 \le \infty$ and $f \in \mathcal{O}(A_{r_1,r_2}(z_0))$. Then, there exists a unique pair of holomorphic functions $f^+ \in \mathcal{O}(B_{r_2}(z_0))$ and $f^- \in \mathcal{O}(\mathbb{C} \setminus \overline{B_{r_1}(z_0)})$ such that*

$$f(z) = f^+(z) + f^-(z), \quad \forall z \in A_{r_1,r_2}(z_0) \quad \textit{and} \quad \lim_{|z|\to\infty} f^-(z) = 0$$

Proof. For any $r_1 < s < r_2$ define $f_s : \mathbb{C} \setminus \partial B_s(z_0) \to \mathbb{C}$ via

$$f_s(z) := \frac{1}{2\pi \mathrm{i}} \int_{\partial B_s(z_0)} \frac{f(\zeta)}{\zeta - z}\, \mathrm{d}\zeta,$$

By Lemma 2.42, f_s is holomorphic. Now define $f^+ : B_{r_2}(z_0) \to \mathbb{C}$ as follows. For a given z choose $r_1 < s < r_2$ such that $|z| < s$ and set $f^+(z) := f_s(z)$. Proposition 3.13 ensures that this definition does not depend on the choice of s. Moreover, it is clear that this defines a holomorphic function. Similarly, we define $f^- : \mathbb{C} \setminus \overline{B_{r_1}(z_0)} \to \mathbb{C}$ as follows. For a given z choose $r_1 < s < r_2$ such that $s < |z|$ and set $f^-(z) := -f_s(z)$. Again, this definition does not depend on the choice of s and f^- is holomorphic.

Now let $z \in A_{r_a,r_2}(z_0)$ and choose s_1, s_2 such that $r_1 < s_1 < |z| < s_2 < r_2$. Then, by Proposition 3.13 we have,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{s_2}(z_0)} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta - \frac{1}{2\pi \mathrm{i}} \int_{\partial B_{s_1}(z_0)} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta = f^+(z) + f^-(z).$$

Fix $r_1 < s < r_2$ and choose $\epsilon > 0$. Now if

$$|z| > \frac{\|f\|_{\partial B_s(z_0)}}{\epsilon} + s + |z_0|,$$

then we have $|f^-(z)| < \epsilon$ by an application of the integral estimate of Proposition 2.7. Thus $\lim_{|z|\to\infty} f^-(z) = 0$.

To see uniqueness suppose there is another pair of holomorphic functions $g^+ \in \mathcal{O}(B_{r_2}(z_0))$ and $g^- \in \mathcal{O}(\mathbb{C} \setminus \overline{B_{r_1}(z_0)})$ with the same properties. Then, $h(z) := f^+(z) - g^+(z)$ defines a holomorphic function on $B_{r_2}(z_0)$. Moreover, for $z \in A_{r_1,r_2}(z_0)$ we also have $h(z) = g^-(z) - f^-(z)$. But the latter are even defined on $\mathbb{C} \setminus \overline{B_{r_1}(z_0)}$. So h extends to an entire function. But, $\lim_{|z|\to\infty} h(z) = \lim_{|z|\to\infty} g^-(z) - \lim_{|z|\to\infty} f^-(z) = 0$. So by Liouville's Theorem (Theorem 2.35) h must be constant and therefore can only be equal to zero. □

Definition 3.15. In the above Theorem, f^+ is called the *regular* part of f while f^- is called the *principal* or *singular* part of f.

Theorem 3.16 (Laurent Series)**.** *Let $z_0 \in \mathbb{C}$ and $0 \le r_1 < r_2$ and $f \in \mathcal{O}(A_{r_1,r_2}(z_0))$. Then, there exist a unique set of coefficients $\{c_n\}_{n\in\mathbb{Z}}$ such that*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z - z_0)^n,$$

where the series converges absolutely for all $z \in A_{r_1,r_2}(z_0)$ and uniformly on $A_{s_1,s_2}(z_0)$, when $r_1 < s_1 < s_2 < r_2$. Also, the coefficients are given by

$$c_n = \frac{1}{2\pi \mathrm{i}} \int_{\partial B_s(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \,\mathrm{d}\zeta,$$

where $r_1 < s < r_2$.

Proof. We use the decomposition $f = f^+ + f^-$ of Theorem 3.14. Define $g \in \mathcal{O}(B_{1/r_1}(0) \setminus \{0\})$ via

$$g(z) := f^-\left(\frac{1}{z} + z_0\right).$$

Since $\lim_{|z|\to\infty} f^-(z) = 0$ it follows that $\lim_{z\to 0} g(z) = 0$. In particular, g has a continuous extension to $B_{1/r_1}(0)$ and thus a holomorphic one by the Riemann Continuation Theorem (Theorem 2.28). Consider its power series expansion

$$g(z) = \sum_{n=1}^{\infty} b_n z^n,$$

which converges pointwise in $B_{1/r_1}(0)$ and uniformly in $B_{1/s_1}(0)$ for any $s_1 > r_1$. Thus

$$f^-(z) = g\left(\frac{1}{z-z_0}\right) = \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

converges pointwise in $\mathbb{C} \setminus \overline{B_{r_1}(z_0)}$ and uniformly on $\mathbb{C} \setminus \overline{B_{s_1}(z_0)}$ for any $s_1 > r_1$. On the other hand, the power series expansion

$$f^+(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

converges pointwise in $B_{r_2}(z_0)$ and uniformly on $B_{s_2}(z_0)$ for any $0 < s_2 < r_2$. Summing both expansions and setting $c_{-n} := b_n$ for all $n \in \mathbb{N}$ yields the Laurent series with the desired properties.

Set $r_1 < s < r_2$. We use the formula

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial B_s(z_0)} (\zeta - z_0)^k \,\mathrm{d}\zeta = \begin{cases} 1 & \text{if } k = -1 \\ 0 & \text{if } k \in \mathbb{Z} \setminus \{-1\} \end{cases}$$

(which follows for example from Theorems 2.22 and 2.43) together with convergence of the Laurent series and interchangeability of limit and integral (Proposition 2.8) to obtain the desired formula for the coefficients c_n. □

Proposition 3.17. *Let $D \subsetneq \mathbb{C}$ be a region, $a \in D$ and $f \in \mathcal{O}(D \setminus \{a\})$. Let $r > 0$ such that $A_{0,r}(a) \subset D$. Let*

$$f(z) = \sum_{n\in\mathbb{Z}} c_n (z-a)^n$$

be the Laurent series for f in $A_{0,r}(a)$. Then,

1. *a is a removable singularity of f iff $c_n = 0$ for all $n < 0$.*

2. *a is a pole of order m of f iff $c_{-m} \neq 0$ and $c_n = 0$ for all $n < -m$.*

3. *a is an essential singularity of f iff there exist infinitely many $n < 0$ such that $c_n \neq 0$.*

Proof. **Exercise.** □

Exercise 32. Let $f \in \mathcal{O}(\mathbb{C} \setminus \{0,1,2\})$ be given by

$$f(z) := \frac{1}{z(z-1)(z-2)}.$$

Give the Laurent series expansion of f in the following regions: $A_{0,1}(0)$, $A_{1,2}(0)$, $A_{2,\infty}(0)$.

Exercise 33. Give the Laurent series expansion of $z \mapsto \exp(1/z)$.

Exercise 34. Investigate how the different types of singularities behave with respect to addition, multiplication, quotienting and composition (whenever the corresponding operations make sense)!

3.4 Meromorphic Functions

Definition 3.18. Let $D \subseteq \mathbb{C}$ be a region and $A \subset D$ a discrete and relatively closed subset. Then, $f \in \mathcal{O}(D \setminus A)$ is called *meromorphic* in D if all points $a \in A$ are either removable singularities or poles of f. The set of meromorphic functions in D is denoted by $\mathcal{M}(D)$.

Proposition 3.19. *Let $D \subseteq \mathbb{C}$ be a region. Then, the set $\mathcal{M}(D)$ forms a vector space over $\mathbb{C}$ and moreover forms a field. That is, sums, scalar multiples, products and quotients of meromorphic functions are meromorphic. (Except the quotient by the zero function.)*

Proof. **Exercise.** □

Exercise 35. Show that the set of rational functions forms a proper subfield of $\mathcal{M}(\mathbb{C})$.

Theorem 3.20 (Argument Principle)**.** *Let $D \subseteq \mathbb{C}$ be a region, $f \in \mathcal{M}(D)$. Suppose $Z \subset D$ is the set of zeros of f and $P \subset D$ is the set of poles of f. Suppose γ is a closed path in $D \setminus (Z \cup P)$ such that $\mathrm{Int}_\gamma \subset D$. Then,*

$$\sum_{z \in Z} N(z)\mathrm{Ind}_\gamma(z) - \sum_{z \in P} N(z)\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z,$$

where $N(z)$ is the order of the zero or pole z.

Proof. **Exercise.**[Hint: Generalize the proof of Theorem 3.4.] □

Theorem 3.21 (Rouché's Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region and $f, g \in \mathcal{M}(D)$. Let $Z_f, Z_g \subset D$ be the sets of zeros of f and g and $P_f, P_g \subset D$ the sets of poles of f and g. Let γ be a closed path such that $|\gamma| \in D \setminus (P_f \cup P_g)$ and $\mathrm{Int}\gamma \subset D$. Suppose that*

$$|f(\zeta) + g(\zeta)| < |f(\zeta)| + |g(\zeta)| \quad \forall \zeta \in |\gamma|.$$

Then,

$$\sum_{z \in Z_f} N(z)\mathrm{Ind}_\gamma(z) - \sum_{z \in P_f} N(z)\mathrm{Ind}_\gamma(z) = \sum_{z \in Z_f} N(z)\mathrm{Ind}_\gamma(z) - \sum_{z \in P_g} N(z)\mathrm{Ind}_\gamma(z),$$

where $N(z)$ denotes the order of the zero or pole z.

Proof. First, note that the inequality also implies $|\gamma| \cap Z_f = \emptyset$ and $|\gamma| \cap Z_g = \emptyset$. Set $U := D \setminus (Z_f \cup Z_g \cup P_f \cup P_g)$ and $h(z) := f(z)/g(z)$ for all $z \in U$. Then, $h \in \mathcal{O}(U)$. Note that the hypothesis is equivalent to the inequality

$$|h(z) + 1| < |h(z)| + 1 \quad \forall z \in |\gamma|.$$

This inequality implies that $h(z)$ cannot be a non-negative real number (since in that case there would be equality). That is, $h(z) \in \mathbb{C} \setminus \mathbb{R}_0^+$ for all $z \in |\gamma|$. But since $|\gamma|$ is compact there must a neighborhood $V \subseteq U$ of $|\gamma|$ such that $h(z) \in \mathbb{C} \setminus \mathbb{R}_0^+$ for all $z \in V$. Now, $\mathbb{C} \setminus \mathbb{R}_0^+$ is star-shaped so that $z \mapsto 1/z$ is integrable there (Corollary 2.15), i.e., has a primitive $l \in \mathcal{O}(\mathbb{C} \setminus \mathbb{R}_0^+)$. ($l$ is in fact a branch of the logarithm.) But $l \circ h \in \mathcal{O}(V)$ is a primitive of $h'/h \in \mathcal{O}(V)$, so the integral of h'/h along $|\gamma|$ vanishes (by Proposition 2.11). This means,

$$0 = \int_\gamma \frac{h'(z)}{h(z)}\,\mathrm{d}z = \int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z - \int_\gamma \frac{g'(z)}{g(z)}\,\mathrm{d}z.$$

The result follows then from Theorem 3.20. □

<u>Exercise</u> 36. Let $D, D' \subseteq \mathbb{C}$ be regions such that $D' \subset D$. Consider the linear map $\mathcal{O}(D) \to \mathcal{O}(D')$ induced by the restriction of functions on D to D'. (a) Show either that this map must be injective or that it cannot be injective. (b) Show either that this map must be surjective or that it cannot be surjective.

Exercise 37. Let $D \subseteq \mathbb{C}$ be a bounded region. Define $\tilde{\mathcal{O}}(D) \subseteq \mathcal{O}(D)$ to be the set of holomorphic functions f on D such that f extends to a holomorphic function on some open neighborhood of $\overline{D}$. Likewise, define $\tilde{\mathcal{M}}(D) \subseteq \mathcal{M}(D)$ to be the set of meromorphic functions f on D such that f extends to a meromorphic function on some neighborhood of $\overline{D}$. (a) Show that $\tilde{\mathcal{O}}(D)$ is a proper vector subspace of $\mathcal{O}(D)$. Likewise, show that $\tilde{\mathcal{M}}(D)$ is a proper subfield of $\mathcal{M}(D)$. (b) Show that $\tilde{\mathcal{M}}(D)$ is the quotient field of $\tilde{\mathcal{O}}(D)$. In other words, show that for every element $f \in \tilde{\mathcal{M}}(D)$ there exist elements $g, h \in \tilde{\mathcal{O}}(D)$ such that $f = g/h$. (c) Comment on the possible problems that would appear if one replaces in this exercise $\tilde{\mathcal{O}}(D)$ with $\mathcal{O}(D)$ and $\tilde{\mathcal{M}}(D)$ with $\mathcal{M}(D)$.

Exercise 38. Let $D \subseteq \mathbb{C}$ be a region such that $\overline{B_1(0)} \subset D$ and $f \in \mathcal{O}(D)$. Suppose $|f(z)| < 1$ for all $z \in \partial B_1(0)$. Show that f has precisely one fixed point in $B_1(0)$.

Exercise 39. Determine the number of zeros (counted with order) of the following functions in the specified domain:

1. $z^5 + \frac{1}{3}z^3 + \frac{1}{4}z^2 + \frac{1}{3}$ in $B_1(0)$ and in $B_{1/2}(0)$.
2. $z^5 + 3z^4 + 9z^3 + 10$ in $B_1(0)$ and $B_2(0)$.
3. $9z^5 + 5z - 3$ in $A_{1/2,5}(0)$.
4. $z^8 + z^7 + 4z^2 - 1$ in $B_1(0)$ and $B_2(0)$.

3.5 Residues

Definition 3.22. Let $a \in \mathbb{C}$ and $0 < r$, $f \in \mathcal{O}(B_r(a) \setminus \{a\})$ and

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z-a)^n$$

the Laurent series of f at a. Then, $\mathrm{Res}(f, a) := c_{-1}$ is called the *residue* of f at a.

Theorem 3.23 (Residue Theorem). *Let $D \subseteq \mathbb{C}$ be a region, $A \subset D$ a discrete and relatively closed subset, and $f \in \mathcal{O}(D \setminus A)$. Let γ be a closed path with $|\gamma| \subset D \setminus A$ and $\mathrm{Int}_\gamma \subset D$. Then,*

$$\sum_{a \in A} \mathrm{Res}(f, a) \mathrm{Ind}_\gamma(a) = \frac{1}{2\pi \mathrm{i}} \int_\gamma f(z)\, \mathrm{d}z.$$

Proof. Define $\tilde{A} := \mathrm{Int}_\gamma \cap A$. This is finite since $\mathrm{Int}_\gamma \cup |\gamma|$ is compact. Thus, suppose $\tilde{A} = \{a_1, \ldots, a_n\}$. Observe that the sum in the statement really only runs over $\tilde{A}$, since the index of the other elements of A vanishes. Now, decompose f into a sum

$$f(z) = f_1(z) + \cdots + f_n(z) + g(z) \quad \forall z \in D \setminus A,$$

where $f_k \in \mathcal{O}(\mathbb{C} \setminus \{a_k\})$ and $g \in \mathcal{O}((D \setminus A) \cup \tilde{A})$ as follows. Let f_1 be the singular part f^- of f at a_1 (according to Theorem 3.14). In particular $\mathrm{Res}(f, a_1) = \mathrm{Res}(f_1, a_1)$. Note that $f - f_1$ has one singularity less than f (the one at a_1) and moreover $\mathrm{Res}(f, a_k) = \mathrm{Res}(f - f_1, a_k)$ for all $k > 1$. Now, take f_2 to be the singular part of $f - f_1$ at a_2 etc. Finally, let $g := f - f_1 - \cdots - f_n$ and notice that g has no singularities in Int_γ left. Note that the integral over g along γ vanishes by Theorem 2.43. Thus, the Theorem reduces to proving the identity,

$$\mathrm{Res}(h, a)\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi\mathrm{i}} \int_\gamma h(z)\,\mathrm{d}z$$

for functions $h \in \mathcal{O}(\mathbb{C} \setminus \{a\})$ such that $\lim_{|z| \to \infty} h(z) = 0$. Consider the Laurent series of h around a,

$$h(z) = \sum_{n=-\infty}^{-1} c_n (z - a)^n.$$

Since this converges uniformly on the compact set $|\gamma|$, we can interchange integration and summation,

$$\int_\gamma h(z)\,\mathrm{d}z = \sum_{n=-\infty}^{-1} c_n \int_\gamma (z - a)^n\,\mathrm{d}z.$$

Now note that $(z - a)^n$ has a primitive if $n \leq -2$, i.e., is then integrable in $\mathbb{C} \setminus \{a\}$. Thus, by Proposition 2.11 its integral vanishes. Hence,

$$\int_\gamma h(z)\,\mathrm{d}z = c_{-1} \int_\gamma (z - a)^{-1}\,\mathrm{d}z = \mathrm{Res}(h, a) 2\pi\mathrm{i}\mathrm{Ind}_\gamma(a).$$

This completes the proof. $\square$

<u>**Exercise**</u> **40.** Let $D \subseteq \mathbb{C}$ be a region and $a \in D$. Let $g, h \in \mathcal{O}(D)$ such that $g(a) \neq 0$ and $h(a) = 0$, but $h'(a) \neq 0$. Show that $f := g/h \in \mathcal{M}(D)$ has a simple pole at a and,

$$\mathrm{Res}(f, a) = \frac{g(a)}{h'(a)}.$$

Exercise 41. Calculate the following integrals:

1. $$\int_0^\infty \frac{x^2}{x^4+x^2+1}\,\mathrm{d}x$$
2. $$\int_0^\infty \frac{\cos(x)-1}{x^2}\,\mathrm{d}x$$
3. $$\int_0^\pi \frac{\cos(2\theta)}{1-2a\cos(\theta)+a^2}\,\mathrm{d}\theta, \quad a^2<1$$
4. $$\int_0^\pi \frac{1}{(a+\cos(\theta))^2}, \quad a>1$$

Exercise 42. Show that the following identities hold:

1. $$\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{2}$$
2. $$\int_0^\infty \frac{1}{(x^2+a^2)^2}\,\mathrm{d}x = \frac{\pi}{4a^3}, \quad a>0$$